1.1 Single Qubit

In Computer Science, we use "bits" to measure the quantity of information. In Quantum Computing, we use "quantum bits", or "qubit" for short.

A $\boxed{\text{qubit}}$, similar to bit, has a state. Two possible states for a qubits are $|0\rangle$ and $|1\rangle$.

Notation "$|\ \rangle$" is called the $\boxed{\text{Dirac notation}}$, and it's a standard notation for states in quantum mechanics.

Needs to point out that, different from bit, a qubit can be in a state <u>other</u> than $|0\rangle$ or $|1\rangle$. It's even possible to form a <u>linear combination</u> of states, often called $\boxed{\text{superpositions}}$:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1.$$

α & β are complex numbers. So, qubit is a vector in two-dimentional complex vetor space. Since we have vector space, natually, $|0\rangle$ & $|1\rangle$ can be treated as $\boxed{\text{computational basis states}}$.

For a bit in conventional computer, we can easily ask our computer to check whether it's 0 or 1. However, for qubit, we cannot exam its quantum state, that is the value of α & β. What we can get is

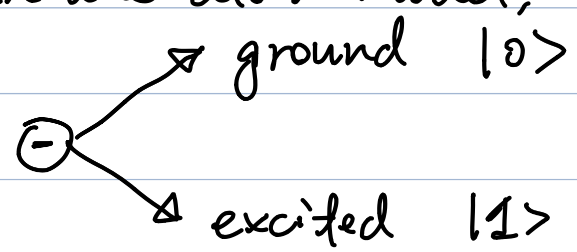
$$\begin{cases} 0 \text{ with } |\alpha|^2 \text{ prob} \\ 1 \text{ with } |\beta|^2 \text{ prob} \end{cases}$$

Naturally, $|\alpha|^2 + |\beta|^2 = 1$.

The reason that we cannot have an accurate observation of quantum state stems from the famous "Uncertainty Principle".

This dichotomy between the unobservation state and the observation we can make lies at the core part of quantum information.

Now, let's talk a little more about the underline physics world. A qubit can be abstracted from the state of a electron. In the atom model, electrons have two states.



By charging electrons with power, say shining lights, it's possible that an electron's state changes from ground, $|0\rangle$, to excited, $|1\rangle$. But if we tune the energy of the light, an electron in ground state initially may jump halfway into a state between $|0\rangle$ & $|1\rangle$.

Let's go back to the mathematical model. We mentioned that any qubit state $|\psi\rangle$ can be expressed using

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

But the complex numbers α & β are not convenient. Besides, we also know that

$$|\alpha|^2 + |\beta|^2 = 1,$$

we can make use of Eular's formula to rewrite $|\psi\rangle$.

let $\alpha = e^{\gamma i}\cos\frac{\theta}{2}$

$\beta = e^{(\gamma+\varphi)i}\sin\frac{\theta}{2}$ $\Rightarrow |\alpha|^2 + |\beta|^2 = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1.$

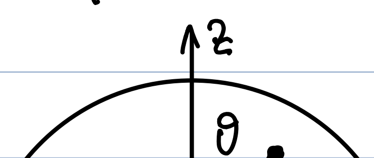
, then,

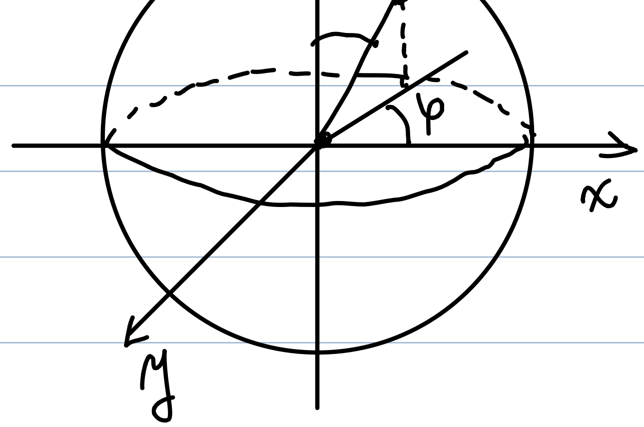
$$|\psi\rangle = e^{\gamma i}\left(\cos\frac{\theta}{2}|0\rangle + e^{\varphi i}\sin\frac{\theta}{2}|1\rangle\right)$$

In the future content, we will show that $e^{\gamma i}$ won't affect the observation, so we can simply use

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{\varphi i}\sin\frac{\theta}{2}|1\rangle$$

θ & φ define a point on the surface of a unit ball.





Since we want a ball (sphere) rather than a semi-ball, we use $\frac{\theta}{2}$ in the subsitution.

The sphere model is called $\boxed{\text{Bloch Sphere}}$. It's a useful tool to visualize a single qubit's state. But, it's also important that the Bloch Sphere model can not be generalized easily to multiple qubits.

Another important point of qubit is that $\boxed{\text{the amount of information a qubit keeps} \leq 1 \text{ bit!}}$. It seems that due to the continuous range of θ & φ, the amount of information should be zero, but the headache fact is that once we measure the state of a qubit, the observation is either 0 or 1. of course, with probility. But if we measure it twice, the second, even the third, forth, ..., observation will always be the same as the first time. Nobody knows why. If you are interested, you can read more about the famous but also notorious experiment Double Seam Interference.

1.2. Mutiple Qubits.

If we have two classical bits, we know that it can have four combinations, 00, 01, 10, 11. For two qubits, the story is similar.

$$|\psi\rangle = \alpha_1 |00\rangle + \alpha_2 |01\rangle + \alpha_3 |10\rangle + \alpha_4 |11\rangle$$

If we measure the qubits, the measurement would be

$$\begin{cases} 00 & \text{w.p. } |\alpha_1|^2 \\ 01 & \text{w.p } |\alpha_2|^2 \\ 10 & \text{w.p } |\alpha_3|^2 \\ 11 & \text{w.p } |\alpha_4|^2 \end{cases}$$

If we only measure the first qubit (or the second). using Bayes' Rule.

If first bit is 1

$$|\psi'\rangle = \frac{\alpha_3 |10\rangle + \alpha_4 |11\rangle}{\sqrt{|\alpha_3|^2 + |\alpha_4|^2}}$$

An important two-qubit state is the Bell state or EPR pair. Many interesting phenomenon are related to this state, such as Qubit Teleportation and Superdense Coding.

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Qubit Teleportation will show up in the next manuscript.